

Spinning black strings in five dimensional Einstein–Gauss-Bonnet gravity

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Abstract

We construct generalizations of the $D = 5$ Kerr black string by including higher curvature corrections to the gravity action in the form of the Gauss-Bonnet density. These uniform black strings satisfy a generalised Smarr relation and share the basic properties of the Einstein gravity solutions. However, they exist only up to a maximal value of the Gauss-Bonnet coupling constant, which depends on the solutions' mass and angular momentum.

1 Introduction

For a spacetime dimension $D > 4$, the Einstein gravity presents a natural generalisation – the so called Lovelock theory, constructed from vielbein, the spin connection and their exterior derivatives without using the Hodge dual, such that the field equations are second order [1], [2]. Following the Ricci scalar, the next order term in the Lovelock hierarchy is the Gauss-Bonnet (GB) one, which contains quadratic powers of the curvature. As discussed in the literature, this term appears as the first curvature stringy correction to general relativity [3, 4], when assuming that the tension of a string is large as compared to the energy scale of other variables. The action of the Einstein-Gauss-Bonnet (EGB) gravity is

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left(R + \frac{\alpha}{4} L_{GB} \right), \quad (1)$$

with

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\sigma\kappa\tau}R^{\mu\sigma\kappa\tau}, \quad (2)$$

where G is Newton's constant, R is the Ricci scalar, g is the determinant of the metric, $R_{\mu\nu}$ is the Ricci tensor, while $R_{\mu\sigma\kappa\tau}$ is the Riemann tensor. The constant α in (1) is the GB coefficient with dimension $(length)^2$ and is positive in the string theory. The variation of the action (1) with respect to the metric tensor results in the EGB equations

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{\alpha}{4}H_{\mu\nu} = 0, \quad (3)$$

where

$$H_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_{\nu}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^{\sigma}_{\nu} + RR_{\mu\nu}) - \frac{1}{2}L_{GB}g_{\mu\nu}, \quad (4)$$

is the Lanczos (or the Gauss-Bonnet) tensor. These equations contain no higher derivatives of the metric tensor than second order and the model has proven to be free of ghost when expanding around flat space.

As expected, inclusion of a GB term in the gravity action leads to a variety of new features (see [5], [6] for recent reviews of the higher order gravity theories and their solutions). However, although the generalization of the spherically symmetric Schwarzschild-Tangherlini solution in EGB theory has been known for quite a long time [7], [8], the issue of solutions with compact extra dimensions is less explored. Black string solutions,

present for $D \geq 5$ spacetime dimensions, are of particular interest, since they exhibit new features that have no analogue in the black hole case.

In the case of Einstein gravity, the simplest solutions of this type are found by trivially extending to D dimensions the vacuum solutions to Einstein equations in $D - 1$ dimensions. These then usually correspond to uniform black strings (UBSs) with horizon topology $S^{D-3} \times S^1$. However, this simple construction does not generically work in the presence of a GB term in the action [9]. The only existing results in the literature on UBSs with a GB term concern the case of static configurations. UBSs in five spacetime dimensions were discussed in [10], as well as their $D > 5$ p -brane generalizations [11]. The results there show the existence of a number of new features in this case, for example the occurrence of a minimal value of the black strings' mass for a given GB parameter α (see also [12]). The extension of the results in [10] for all dimensions between five and ten was given in [13].

The purpose of this work is to construct spinning generalizations of the known UBSs in EGB theory. For simplicity, we shall restrict to the case of five spacetime dimensions¹. By solving numerically the field equations, we show that the $\alpha = 0$ solution (*i.e.* the Kerr black string) admits generalizations with a GB term and discuss the new features which occur in this case.

2 The model

2.1 Black strings in EGB theory: general formalism

In this work we are interested in spinning solutions approaching asymptotically the four dimensional Minkowski-space times a circle, $\mathcal{M}_4 \times S^1$. The line element of this background is

$$ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\Omega_2^2, \quad (5)$$

where the direction z is periodic with period L , r and t are the radial and time coordinates, respectively, while $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the unit metric on S^2 .

The physical quantities of a spinning configuration that can be measured asymptotically far away in the transverse space are the mass M , the tension \mathcal{T} in the direction of the circle, and the angular momentum J . Similar to Einstein gravity, these quantities are defined in terms of three constants c_t, c_z and c_ϕ which enter the asymptotics of the metric functions

$$g_{tt} \simeq -1 + \frac{c_t}{r}, \quad g_{zz} \simeq 1 + \frac{c_z}{r}, \quad g_{\phi t} \simeq \frac{c_\phi \sin^2 \theta}{r}. \quad (6)$$

The mass, tension and angular momentum of a spinning black string solution are given by²

$$M = \frac{V_2 L}{16\pi G} [2c_t - c_z], \quad \mathcal{T} = \frac{V_2}{16\pi G} [c_t - 2c_z], \quad J = \frac{V_2 L}{8\pi G} c_\phi, \quad (7)$$

where $V_2 = 4\pi$ is the area of the unit S^2 sphere.

Similar to the static case, one can also define a relative tension n (also called the relative binding energy)

$$n = \frac{\mathcal{T} L}{M} = \frac{c_t - 2c_z}{2c_t - c_z}, \quad (8)$$

which measures how large the tension is relative to the mass. Uniform string solutions in vacuum Einstein gravity have $c_z = 0$ and thus a relative tension $n = 1/2$. However, c_z does not vanish in the presence of a GB term, which leads to a relative tension $n \neq 1/2$ even in the static case [13].

The Hawking temperature of the solutions is given by

$$T_H = \frac{\kappa}{2\pi}, \quad (9)$$

¹The case $D = 5$ is interesting from yet another point of view, since the GB term appears there in the low-energy effective action for the compactification of the M -theory on a Calabi-Yau threefold [14].

²For discussions of the computation of charges in EGB theory without a cosmological constant, see [15].

with κ the surface gravity. The general results in [16] show that the entropy of a black object (*i.e.* also of a black string) in EGB theory can be written as an integral over the event horizon,

$$S = \frac{1}{4G} \int_{\Sigma_h} d^3x \sqrt{h} (1 + \frac{\alpha}{2} \tilde{R}), \quad (10)$$

where h is the determinant of the induced metric on the horizon and \tilde{R} is the event horizon curvature.

The solutions should obey the first law of thermodynamics, which for spinning solutions contains an extra work term:

$$dM = T_H dS + \mathcal{T} dL + \Omega_H dJ, \quad (11)$$

where Ω_H (the thermodynamic variable conjugate to J) is the event horizon velocity.

Interestingly, one can show that for solutions without a dependence on the extra-dimensions z , the event horizon quantities T_H , S , Ω_H and the global charges M , \mathcal{T} are related through the simple Smarr mass formula³

$$M - \mathcal{T}L = T_H S + \Omega_H J. \quad (12)$$

An interesting feature of EGB gravity is the presence of two branches of static solutions, distinguished by their behaviour for $\alpha \rightarrow 0$ [7]. In this work we shall restrict our analysis to rotating UBSs whose static limit corresponds to the branch of static solutions with a well defined Einstein gravity limit.

2.2 The metric ansatz and boundary conditions

Our solutions possess three Killing vectors ∂_t , ∂_φ and ∂_z and are constructed within the following metric ansatz⁴

$$ds^2 = -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2 \theta \left(d\varphi - \frac{\omega}{r} dt \right)^2 + p dz^2, \quad (13)$$

where f , m , l , p and ω are functions of r and θ , only. The event horizon of these stationary black holes resides at a surface of constant radial coordinate $r = r_H$, and is characterized by the condition $f(r_H) = 0$.

At the horizon we impose the boundary conditions

$$f|_{r=r_H} = m|_{r=r_H} = l|_{r=r_H} = 0, \quad \omega|_{r=r_H} = \Omega_H r_H, \quad \partial_r p|_{r=r_H} = 0. \quad (14)$$

The boundary conditions at infinity,

$$f|_{r=\infty} = m|_{r=\infty} = l|_{r=\infty} = p|_{r=\infty} = 1, \quad \omega|_{r=\infty} = 0, \quad (15)$$

ensure that the solutions approach asymptotically the Kaluza-Klein background (5). Axial symmetry and regularity impose the boundary conditions on the symmetry axis ($\theta = 0$),

$$\partial_\theta f|_{\theta=0} = \partial_\theta l|_{\theta=0} = \partial_\theta m|_{\theta=0} = \partial_\theta \omega|_{\theta=0} = \partial_\theta p|_{\theta=0} = 0, \quad (16)$$

and, for solutions with parity reflection symmetry (the case in this work), agree with the boundary conditions on the $\theta = \pi/2$ -axis. The absence of conical singularities implies also $m = l$ at $\theta = 0$.

Expansion near the horizon in $\delta = (r - r_H)/r_H$ yields to lowest order $f = \delta^2 f_2(\theta)$, $m = \delta^2 m_2(\theta)$, $l = \delta^2 l_2(\theta)$, $\omega = \Omega_H r_H (1 + \delta)$ and $p = p_0(\theta) + \delta^2 p_2(\theta)$. The metric of a spatial cross-section of the horizon reads

$$d\sigma^2 = \frac{m_2(\theta)}{f_2(\theta)} r_H^2 d\theta^2 + \frac{l_2(\theta)}{f_2(\theta)} r_H^2 \sin^2 \theta d\varphi^2 + p_0(\theta) dz^2, \quad (17)$$

³This relation is obtained by starting from the Komar expressions, and making use of the equations of motion and the expansion of the solutions at the horizon and at infinity.

⁴ The choice in (13) of a conformal gauge for the (r, θ) sector of the metric, instead of the usual choice for Boyer-Lindquist coordinates, leads to a more stable numerical scheme. Also, for $\omega = 0$, this line element describes static UBSs in an 'isotropic' coordinate system (see the discussion in Section 4 of Ref. [19]).

the computation of the entropy from (10) being straightforward,

$$S = \frac{\pi L}{2G} \int_0^\pi d\theta \left\{ r_H^2 \sin \theta \frac{\sqrt{l_2 m_2 p_0}}{f_2} + \frac{\alpha}{2} \sqrt{\frac{l_2 p_2}{m_2}} \left[\sin \theta \left(2 - \frac{l_2''}{l_2} + \frac{m_2''}{m_2} - \frac{p_0''}{p_0} + \frac{3l_2' m_2'}{4l_2 m_2} + \frac{m_2' p_0'}{m_2 p_0} \right. \right. \right. \\ \left. \left. \left. - \frac{l_2' p_0'}{2l_2 p_0} + \frac{l_2'^2}{2l_2^2} - \frac{3m_2'^2}{4m_2^2} + \frac{p_0'^2}{2p_0^2} \right) + \cos \theta \left(\frac{3m_2'}{2m_2} - \frac{2l_2'}{l_2} - \frac{p_0'}{p_0} \right) \right] \right\}, \quad (18)$$

(where a prime denotes $d/d\theta$). Also, since f_2 , l_2 , m_2 and p_0 are strictly positive and finite for all values of θ , and z is a periodic coordinate, it is obvious that the solutions have an $S^2 \times S^1$ event horizon topology. The Hawking temperature T_H of the black strings is

$$T_H = \frac{1}{2\pi r_H} \frac{f_2(\theta)}{\sqrt{m_2(\theta)}}, \quad (19)$$

the field equation $E_\theta^r = 0$ implying that the surface gravity is indeed constant on the horizon. For completeness, we mention that the mass, tension and angular momentum are read from the asymptotic expansion (6), with $g_{tt} = -f$, $g_{zz} = p$, $g_{\phi t} = -\omega r \sin^2 \theta / f$.

2.3 The equations and the Kerr black string

The scarcity of exact solutions is a generic feature of EGB theory⁵. For example, even in the static case, no closed form black string solution could be found within a nonperturbative approach. Therefore we rely on numerical methods also on constructing spinning UBSs.

The solutions in this work are found by using an approach originally proposed in [18] for $D = 4$ solutions of Einstein gravity coupled with other matter fields, which has been generalized in [19] to static solutions of the $D = 5$ EGB theory.

The equations for the functions $\mathcal{F}_i = (f, l, m, \omega, p)$ we employ in the numerics, are found by using a suitable combination of the EGB equations, $E_t^t = 0$, $E_r^r + E_\theta^\theta = 0$, $E_\phi^\phi = 0$, $E_z^z = 0$ and $E_\phi^t = 0$, which diagonalizes the Einstein tensor *w.r.t.* $\nabla^2 \mathcal{F}_i$ (where $\nabla^2 = \partial_{rr} + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_{\theta\theta}$). The remaining equations $E_\theta^r = 0$, $E_r^r - E_\theta^\theta = 0$ yield two constraints. Following [20], one can show that the identities $\nabla_\mu E^{\mu r} = 0$ and $\nabla_\mu E^{\mu \theta} = 0$, imply the Cauchy-Riemann relations

$$\partial_{\bar{r}} \mathcal{P}_2 + \partial_\theta \mathcal{P}_1 = 0, \quad \partial_{\bar{r}} \mathcal{P}_1 - \partial_\theta \mathcal{P}_2 = 0, \quad (20)$$

with $\mathcal{P}_1 = \sqrt{-g} E_\theta^r$, $\mathcal{P}_2 = \sqrt{-g} (E_r^r - E_\theta^\theta)/2$ and $d\bar{r} = \frac{dr}{r}$. Therefore the weighted constraints still satisfy Laplace equations, and the constraints are fulfilled, when one of them is satisfied on the boundary and the other at a single point [20].

The resulting set of five second order coupled non-linear partial differential equations⁶ for the functions \mathcal{F}_i is solved numerically, subject to the boundary conditions (14)-(16), employing a compactified coordinate⁷ $x = 1 - r_H/r$, which leads to a rectangular shape for the domain of integration, $0 \leq x \leq 1$, $0 \leq \theta \leq \pi/2$. The numerical calculations are based on the Newton-Raphson method and are performed with help of the program FIDISOL/CADSOL [21], which provides also an error estimate for each unknown function. For the solutions in this work, the typical numerical error for the functions is estimated to be lower than 10^{-3} . The Smarr relation (12) provides a further test of the numerical accuracy.

In this approach, one provides the input parameters $(\alpha; r_H, \Omega_H) \geq 0$. The quantities of interest are computed from the numerical output (for example, the mass M , tension \mathcal{T} and angular momentum J are extracted from the asymptotic expressions (6)).

⁵In fact, for $D = 5$, the only solution known in closed form corresponds to the generalization of the Schwarzschild-Tangherlini black hole found in [7], [8] (however, see also the spinning configurations with a negative cosmological constant in the Section V of [17]).

⁶Due to the GB contribution, these equations are much more complicated than in the case of Einstein gravity (with more than 100 terms each equation). Then we shall not present them here.

⁷Therefore we restrict the numerical integration to the region outside the horizon, $r \geq r_H$.

The equations satisfied by the metric functions are invariant under the following rescaling:

$$\alpha \rightarrow \lambda^2 \alpha, \quad r \rightarrow \lambda r. \quad (21)$$

Then a dimensionless relevant parameter can be defined according to $\beta = \alpha/\lambda^2$, where λ is some length scale. Following [11], [13], we have found it convenient to choose λ as the horizon radius r_H of the black string and thus to define

$$\beta \equiv \frac{\alpha}{r_H^2}. \quad (22)$$

Also, for UBS solutions, the period L of the z -direction is an arbitrary positive constant and plays no role in our results. Then, to simplify the relations, we set $L = G = 1$ in all results below (*i.e.* one considers the values of M, S and J per unit length of the extra-dimension).

The Kerr black string is recovered for $\alpha = 0$ and has $g_{zz} = p(r, \theta) = 1$, the expression of the other metric functions (for the metric ansatz (13)) being

$$f = \left(1 - \frac{r_H^2}{r^2}\right)^2 \frac{F_1}{F_2}, \quad l = \left(1 - \frac{r_H^2}{r^2}\right)^2, \quad m = \left(1 - \frac{r_H^2}{r^2}\right)^2 \frac{F_1}{F_2}, \quad \omega = \frac{2M\sqrt{M^2 - 4r_H^2}}{r^2} \left(1 + \frac{M}{r} + \frac{r_H^2}{r^2}\right), \quad (23)$$

where

$$F_1 = \frac{2M^2}{r^2} + \left(1 - \frac{r_H^2}{r^2}\right)^2 + \frac{2M}{r} \left(1 + \frac{r_H^2}{r^2}\right) - \frac{M^2 - 4r_H^2}{r^2} \sin^2 \theta, \\ F_2 = \left(\frac{2M^2}{r^2} + \left(1 - \frac{r_H^2}{r^2}\right)^2 + \frac{2M}{r} \left(1 + \frac{r_H^2}{r^2}\right)\right)^2 - \left(1 - \frac{r_H^2}{r^2}\right)^2 \frac{M^2 - 4r_H^2}{r^2} \sin^2 \theta.$$

The value of the horizon velocity (which enters the boundary conditions at $r = r_H$) is expressed in terms of mass and event horizon radius as $\Omega_H = \frac{\sqrt{M^2 - 4r_H^2}}{2M^2 + 4Mr_H}$, with $M \geq 2r_H$. The entropy, angular momentum and the Hawking temperature of the Kerr UBS are given by $S = 2\pi M(M + 2r_H)$, $J = M\sqrt{M^2 - 4r_H^2}$ and $T_H = \frac{1}{4\pi M} \frac{1}{1 + \frac{M}{2r_H}}$, respectively.

In studying the solutions' properties, it is convenient to work with reduced dimensionless quantities as follows:

$$t_H = 8\pi T_H M, \quad j = \frac{J}{M^2}, \quad a_H = \frac{1}{16\pi} \frac{A_H}{M^2}, \quad s = \frac{1}{4\pi} \frac{S}{M^2}, \quad (24)$$

the length scale being fixed in this case by the mass of the solutions. For the solutions without a GB term, one finds from (23) the simple relations $a_H = \frac{1}{2}(1 + \sqrt{1 - j^2})$, $t_H = \frac{2\sqrt{1 - j^2}}{1 + \sqrt{1 - j^2}}$, $s = \frac{1}{2}(1 + \sqrt{1 - j^2})$.

Alternatively, following [10], one can scale all quantities with respect to α , taking into account the corresponding dimensions (*e.g.* $S \sim (length)^2$, $\Omega_H \sim (length)^{-1}$ etc).

3 The results

3.1 The static black strings

Before discussing the spinning UBSs, let us briefly review the situation in the static case. For the metric ansatz (13), these solutions are found in the limit $\omega = 0$ and have $l = m$, with f, m, p functions of r only. The static UBSs were studied within a nonperturbative approach in [10], [13] and [19]. The numerical results there show the existence, for a given value of r_H , of a maximal value of α , with⁸ $\beta^{(max)} = \alpha^{(max)}/r_H^2 \simeq 5.8$.

⁸Note that the results in [10] and [13] were found for a Schwarzschild-like coordinate system. The value of the event horizon radius in that case differs from r_H for the 'isotropic' line-element (13), which translates into a different maximal value of the parameter β .

Since for $\alpha > 0$ there is a finite minimal value of the horizon radius, this entails the existence of a minimal value of the mass⁹ for a given GB coupling constant α , a property which is inherited by the static EGB black rings approaching asymptotically the \mathcal{M}_5 background [19]. This strongly contrasts with the picture found for EGB black holes with an S^3 topology of the horizon, where α takes arbitrary values.

Interestingly, the static UBSs admit an analytic expression as a power series in α around the Einstein gravity solution. The perturbative solution reads

$$f(r) = f_0(r) \left(1 + \sum_{k=1}^{\infty} \alpha^k f_k(r) \right), \quad m(r) = m_0(r) \left(1 + \sum_{k=1}^{\infty} \alpha^k m_k(r) \right), \quad p(r) = 1 + \sum_{k=1}^{\infty} \alpha^k p_k(r), \quad (25)$$

with $f_0 = \left(\frac{1 - \frac{r_H}{r}}{1 + \frac{r_H}{r}} \right)^2$, $m_0 = (1 - \frac{r_H^2}{r^2})^2$, the metric functions for the UBS solution in Einstein gravity. One finds *e.g.* for the first order solution

$$f_1(r) = -\frac{1}{2r^2(1 + \frac{r_H}{r})^6} \left((1 - \frac{r_H}{r})^2 + \frac{44r_H}{9r} + \frac{r}{6r_H} (1 - \frac{r_H^2}{r^2})^2 \right) \quad \text{and} \quad m_1(r) = -p_1(r) = 2f_1(r). \quad (26)$$

The expression of the solution becomes very complicated for higher values of k and we shall not give it here.

Once the (perturbative) solution is known, it is straightforward to extract the relevant global quantities. One finds in this way that, to second order in α , the following relations hold (with $M_0 = 2r_H$, $\mathcal{T}_0 = r_H$, $S_0 = 16\pi r_H^2$ and $T_H^0 = \frac{1}{16\pi r_H}$ the mass, tension, entropy and Hawking temperature in Einstein gravity):

$$M = M_0 \left(1 + \frac{1153\beta^2}{7096320} \right), \quad \mathcal{T} = \mathcal{T}_0 \left(1 - \frac{\beta}{16} - \frac{1129\beta^2}{1182720} \right), \quad S = S_0 \left(1 + \frac{\beta}{16} + \frac{14557\beta^2}{7096320} \right), \quad T_H = T_H^0 \left(1 + \frac{1153\beta^2}{7096320} \right), \quad (27)$$

which provides a reasonable approximation for the (numerical) nonperturbative results. Inclusion of higher order terms in (27) does not change this pattern: the mass, entropy and temperature increase with β , while the tension decreases. At the same time, the linear terms in β are absent in the expressions of M and T_H . As a result, the mass and the temperature of a static EGB black string with a given event horizon radius r_H do not change significantly with the GB parameter α .

3.2 Spinning solutions

In principle, the slowly rotating UBS can be constructed in closed form, by taking the perturbative solution (25) for the static background. For example, to lowest order in α , the expression of the metric function associated with rotation is $\omega = \frac{a(1 + \alpha f_1)}{r^2(1 + \frac{r_H}{r})^6}$, with a the small rotation parameter.

This is linear in the perturbation parameter a , while the other functions remain unchanged to this order in a . However, this approach has some obvious limitations and we shall not pursue it here.

The nonperturbative solutions are found by directly solving the EGB equations for the functions \mathcal{F}_i without any approximation. As expected, we have found numerical evidence that the spinning Einstein gravity solution (23) also admits generalizations with a GB term. These solutions are found by starting with the Kerr metric (with given r_H, Ω_H) as the initial guess, and slowly increasing the value of α . The iterations converge, and repeating the procedure one obtains in this way solutions with large α .

For all solutions we have found, the metric functions \mathcal{F}_i and their first and second derivatives with respect to r and θ have smooth profiles, which leads to finite curvature invariants on the full domain of integration, in particular on the event horizon. The shape of the functions f, l, m and ω is similar to the $\alpha = 0$ case, the maximal deviation from the Einstein gravity profiles being around the horizon. As expected, $\alpha \neq 0$ leads to a metric function $g_{zz} \neq 1$, which in the rotating case, possesses a nontrivial angular dependence, see Figure 1 (left).

⁹Note that this feature is absent for black strings in more than five dimensions [13].

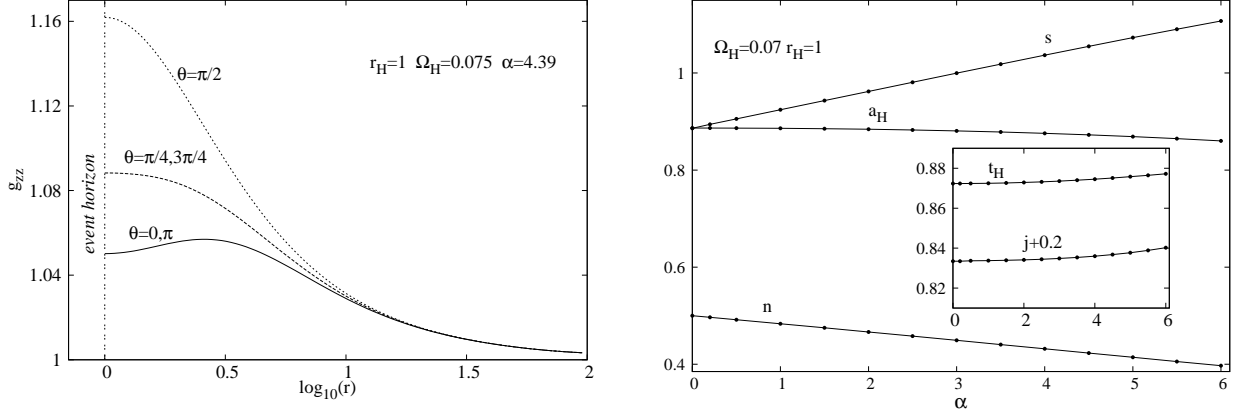


Figure 1. *Left.* The metric function g_{zz} is shown as a function of r and several values of θ for a typical spinning black string in EGB theory. *Right.* A number of reduced parameters are shown as a function of α for a set of black strings with fixed event horizon velocity Ω_H and fixed event horizon radius r_H . Here and in Figures 2, 3, the dots represent the data points while the curves are obtained by spline-interpolation. Also, for all data displayed in this work we set $L = G = 1$.

The general pattern is, however, quite complicated, and depends on the value of the parameter α . As one can see in Figure 1 (right), for given (r_H, Ω_H) , the relative tension n and scaled horizon area a_H decrease with α , while the scaled temperature t_H , entropy s and angular momentum j increase. The picture there seems to be generic and has been recovered for other values of (r_H, Ω_H) .

For given values of the horizon input data (r_H, Ω_H) , we have noticed the existence of a maximal value of the GB parameter α . This translates into a maximal value of the ratio α/M^2 for a given value of the reduced angular momentum $j = J/M^2$. For example, for $j = 0$, one finds $\alpha/M^2 < 1.31$. However, in the spinning case, it is rather difficult to provide such estimates, since both M and J are output parameters and cannot easily be kept fixed.

As $\beta^{(max)} = \alpha^{(max)}/r_H^2$ is approached, the numerical process fails to converge, although no singular behaviour is found there. The technical reason which causes the solutions to cease to exist at $\beta^{(max)}$ is similar to the static case (see *e.g.* the discussion in [19]), and can be seen in the horizon expansion of the metric functions. One finds that, for given (r_H, Ω_H) , the roots of a quadratic equation in the horizon parameters p_0 , m_2 , f_2 cease to be real at $\beta^{(max)}$. We mention that the same behaviour has been noticed for other non-spherically symmetric solutions with a GB term in the action, see [19], [22].

However, for the allowed range of $\beta = \alpha/r_H^2$, the overall picture is rather similar to the case of Einstein gravity, any static black string admitting rotating generalizations. Here it is instructive to keep fixed the parameter β and to study the effects of an increasing event horizon velocity on the properties of UBSs (these solutions are found by starting with the static solutions in [10] (written, however, in the 'isotropic' coordinate system (13)) and slowly increasing the event horizon velocity Ω_H). Some numerical results in this case are shown in Figures 2, 3.

When increasing Ω_H from zero, while keeping (r_H, α) fixed, a branch of spinning UBS solutions forms, the lower branch. It extends up to a maximal value of Ω_H , where an upper branch emerges and bends backwards towards $\Omega_H = 0$. The maximal value of Ω_H depends on (r_H, α) , with $\Omega_H^{(max)} = \frac{1}{2r_H} \frac{\sqrt{2}}{3+\sqrt{5}}$ for $\alpha = 0$. Our results show that the value of $\Omega_H^{(max)}$ slowly decreases with β by a simple scaling. Along both branches, the mass, tension, entropy and angular momentum continuously increase¹⁰. Interestingly, the relative tension n increases also with the angular momentum (see Figure 2 (right)), and appears to approach asymptotically

¹⁰We emphasize that the existence of two branches of solutions in terms of Ω_H for given r_H is a result of using an 'isotropic' coordinate system in (13), and it occurs already for the Kerr UBS.

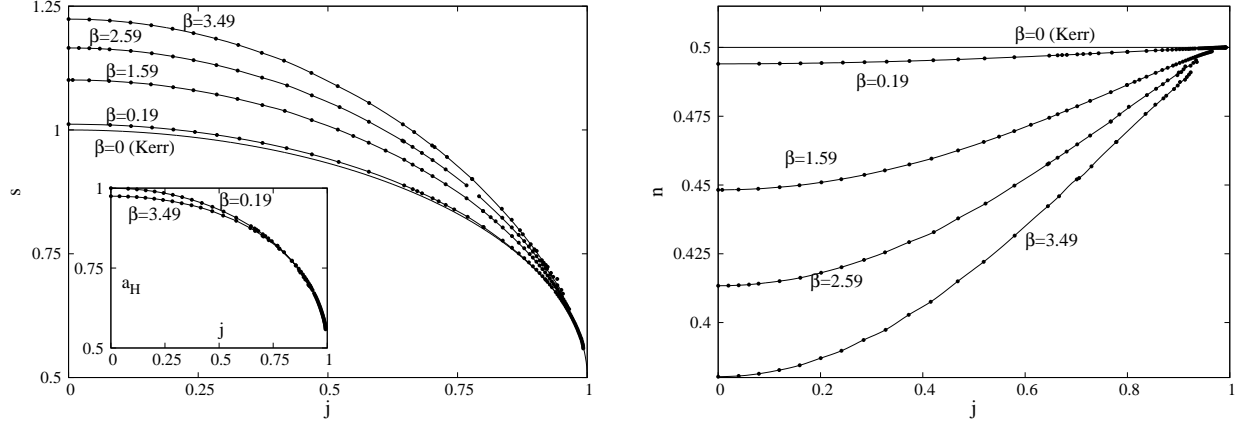


Figure 2. The reduced entropy $s = \frac{1}{4\pi} \frac{S}{M^2}$ and area $a_H = \frac{1}{16\pi} \frac{A_H}{M^2}$ (left) and the relative tension $n = \mathcal{T}L/M$ (right) are plotted *vs.* the reduced angular momentum $j = J/M^2$ for several values of the parameter $\beta = \alpha/r_H^2$.

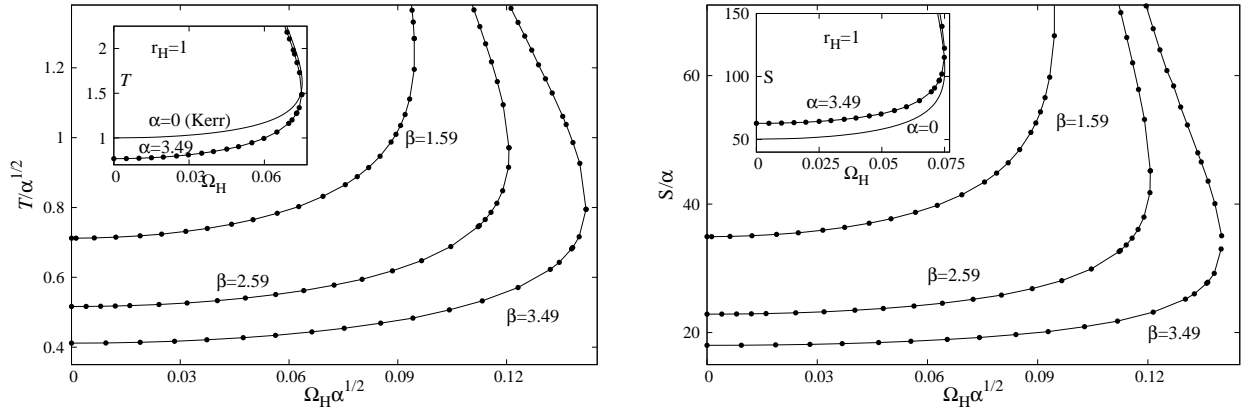


Figure 3. The tension and the entropy are plotted *vs.* the angular momentum velocity for several values of the parameter $\beta = \alpha/r_H^2$, the quantities being given in units of the Gauss-Bonnet constant α . The insets show a comparison between the picture for Einstein and Einstein-Gauss-Bonnet gravity solutions with the same value of the horizon radius (unscaled quantities).

the Kerr value $n = 1/2$ for solutions with $\Omega_H \rightarrow 0$ on the upper branch (*i.e.* $c_z/c_t \rightarrow 0$ in that limit).

Also, we have noticed that, for a given β , the mass and Hawking temperature have only a small deviation from the corresponding values in the Einstein gravity case, while the angular momentum, entropy and tension change significantly. We expect that the explanation of this behaviour would be similar to that found in the static case, namely that no terms linear in α will enter the expressions of M and T_H in the rotating generalization of the perturbative result (27).

3.3 The issue of extremal black strings

For all considered values of $\beta = \alpha/r_H^2$, the numerical iteration fails to converge for solutions on the second branch with small values of Ω_H . In that limit, the Hawking temperature takes very small values, which suggests that the limit $\Omega_H \rightarrow 0$ corresponds to an extremal configuration. For example, the family of solutions with $\alpha = 0$ ends at the extremal Kerr UBS, which precisely saturates the Kerr bound for the scaled

angular momentum.

A study of the extremal UBSs would require a different metric ansatz than (13) and is beyond the purposes of this work. However, we argue that, different from the $\alpha = 0$ extremal Kerr solution, the extremal UBSs with GB corrections are likely to not represent regular configurations. This is supported by our results when attempting to construct the corresponding near-horizon geometries with an isometry group $SO(2, 1) \times U(1) \times U(1)$.

There, following the usual ansatz in the literature (see *e.g.* [23]) we consider the line element

$$ds^2 = v_1(\theta) \left(-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} + \bar{\beta}^2 d\theta^2 \right) + \bar{\beta}^2 v_2(\theta) (d\phi + K \rho dt)^2 + v_3(\theta) dz^2, \quad (28)$$

where $0 \leq \rho < \infty$, $0 \leq \theta \leq \pi$, and $\bar{\beta}$, K are real parameters. The above line element describes the neighbourhood of the event horizon of an extremal UBS (and will be an attractor for the full bulk solutions).

Within this ansatz, the EGB equations (3) result in a set of coupled nonlinear ordinary differential equations. For $\alpha = 0$, the Einstein gravity solution is recovered, with [24]

$$K = \bar{\beta} = 1, \quad v_1 = \frac{J}{32\pi} (3 + 2 \cos 2\theta), \quad v_2 = \frac{J}{4\pi} \frac{\sin^2 \theta}{(1 + \cos^2 \theta)}, \quad v_3 = 1, \quad (29)$$

and $J > 0$ an integration constant.

Unfortunately, no closed form solution could be found in the presence of a GB term. Therefore, as a first step, we have considered a perturbative solution in α of the EGB equations around the above Einstein gravity configuration, with

$$v_1(\theta) = v_{10}(\theta) + \alpha v_{11}(\theta) + O(\alpha)^2, \quad v_2(\theta) = v_{20}(\theta) + \alpha v_{21}(\theta) + O(\alpha)^2, \quad v_3(\theta) = 1 + \alpha v_{31}(\theta) + O(\alpha)^2, \quad (30)$$

and $\bar{\beta} = 1 + \bar{\beta}_1 \alpha + O(\alpha)^2$ (note that one can set $K = 1$ without any loss of generality). Then a straightforward computation shows that the functions $v_{i1}(\theta)$ cannot be regular at both poles of the sphere. For example, the expression for the first order correction to the metric function g_{zz} is

$$v_{31}(\theta) = \frac{64\pi}{3J} \left(-\frac{2(11 + 20 \cos 2\theta + \cos 4\theta)}{(3 + \cos 2\theta)^3} + \log \frac{2(1 + \cos^2 \theta)}{\sin^2 \theta} \right) + c_1 \log(\tan^2 \frac{\theta}{2}). \quad (31)$$

One can see that, for any choice of the arbitrary constant c_1 , the function g_{zz} cannot be regular both at $\theta = 0$ and $\theta = \pi$. A similar result is found when considering higher orders in the expansion (30).

Of course, regular solutions without a smooth Einstein gravity limit are not ruled out by the above argument. Therefore, we have also tried to solve non-perturbatively the set of four EGB equations with suitable boundary conditions at $\theta = 0, \pi$. However, the numerical iteration failed to converge for any finite value of α . Thus we conclude that the extremal black string solutions with a regular horizon are unlikely to exist in EGB theory.

4 Further remarks

In this work we have initiated a preliminary investigation of the influence of the higher derivative terms in the gravity action on the properties of spinning black strings in $D = 5$ spacetime dimensions. Our results give numerical evidence that the well-known Kerr solution in Einstein gravity admits generalizations with a GB term. Similar to the static case, these UBSs exist up to a maximal value of the GB coupling constant α which depends on the event horizon radius and event horizon velocity. Also, we have noticed that the angular velocity reduces the relative tension of the solutions, which approaches (for fast rotating black strings) the Einstein gravity value $n = 1/2$. However, perhaps the most interesting new feature here is that the GB term strongly affects the properties of the extremal black strings, and seems to lead to some unphysical features of these configurations.

We also note an effective violation of the weak energy condition by the UBS solutions of the EGB model. Here, following [19], we write the EGB equations (3) as ‘modified’ Einstein equations, with an effective stress tensor that involves the gravitational field

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{\alpha}{4}H_{\mu\nu} = T_{\mu\nu}. \quad (32)$$

Therefore, from some point of view, the quantity $\alpha H_t^t = -G_t^t$ corresponds to a local ‘*effective energy density*’. We have found numerical evidence that this quantity takes negative values in some region close to the horizon. Moreover, this region expands as the angular momentum increases and the Hawking temperature decreases. It would be interesting to get a deeper understanding of these aspects, preferably based on some global techniques.

One should also remark that since the solutions in this work are without a dependence on the extra-dimension z , they can also be interpreted as black holes in a EGB-dilaton theory in four dimensions. The action of the $D = 4$ model is found by doing a reduction with respect to the Killing vector $\partial/\partial z$ for a generic metric ansatz

$$ds^2 = e^{-\frac{\Phi}{\sqrt{3}}} g_{\mu\nu}^{(4)} dx^\mu dx^\nu + e^{\frac{2\Phi}{\sqrt{3}}} dz^2, \quad (33)$$

(*i.e.* with $g_{zz} = p(r, \theta) = e^{\frac{2\Phi}{\sqrt{3}}}$) and reads (see *e.g.* [10])

$$I = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g^{(4)}} \left[R^{(4)} - \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{\alpha}{4} e^\Phi \left(L_{GB}^{(4)} + \frac{4}{3} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{3\sqrt{3}} (\nabla^2 \Phi) (\partial_\mu \Phi \partial^\mu \Phi)^2 \right) \right]. \quad (34)$$

The line element of the corresponding four-dimensional spinning black holes will be

$$ds_4^2 = g_{\mu\nu}^{(4)} dx^\mu dx^\nu = -\hat{f} dt^2 + \frac{\hat{m}}{\hat{f}} (dr^2 + r^2 d\theta^2) + \frac{\hat{l}}{\hat{f}} r^2 \sin^2 \theta \left(d\varphi - \frac{\hat{\omega}}{r} dt \right)^2, \quad (35)$$

with $\hat{f} = f\sqrt{p}$, $\hat{m} = mp$, $\hat{l} = lp$ and $\hat{\omega} = \omega$ (where f, l, m, ω and p are the metric functions in the five-dimensional line-element (13)). The properties of these solutions result straightforwardly from those of the $D = 5$ black strings discussed in this work.

One should mention that spinning black holes of a simplified $D = 4$ EGB-dilaton model containing only the first three terms in (34) (*i.e.* with a standard kinetic term only for the dilaton), and a different value of the dilaton coupling constant, have been discussed recently in [22]. As expected, they present many common features with the solutions in this work, in particular the extremal limiting configurations being singular in both cases.

Similar to the $\alpha = 0$ case, the generalizations with a $U(1)$ field of the $D = 4$ spinning black holes (35) can be generated by boosting the $D = 5$ UBSs in the fifth direction, $z = \cosh \gamma Z + \sinh \gamma \tau$, $t = \sinh \gamma Z + \cosh \gamma \tau$, with γ an arbitrary parameter. Then the dimensional reduction of a UBS configuration along the Z -direction provides new solutions in a $D = 4$ EGB- $U(1)$ -dilaton theory, generalizing the well-known dilatonic Kerr-Newman black holes in [25]. In principle, based on the results in this work, one can obtain a complete description of these solutions. However, one should remark that due to the presence of the GB term in $D = 5$, the action of this four dimensional model has a very complicated and rather exotic form, with non-standard terms for the dilaton and the $U(1)$ fields (see *e.g.* the Appendix A in Ref. [26]).

As avenues for further research, it would be interesting to extend the solutions in this work by adding $n > 1$ extra-dimensions (“black branes”). Based on the results in [11], we expect these configurations to retain the basic features of the black strings studied here. Another possible direction would be to construct spinning generalizations of the $D > 5$ static EGB black strings discussed in [13] (*i.e.* generalizations of the Myers-Perry black strings), in which case we expect a different pattern of the solutions.

We hope to return with a systematic study of these aspects in a future publication.

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